

# On the existence of connecting orbits for critical values of the energy

Giorgio Fusco,<sup>\*</sup> Giovanni F. Gronchi,<sup>†</sup> Matteo Novaga<sup>‡</sup>

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## Abstract

We consider an open connected set  $\Omega$  and a smooth potential  $U$  which is positive in  $\Omega$  and vanishes on  $\partial\Omega$ . We study the existence of orbits of the mechanical system

$$\ddot{u} = U_x(u),$$

that connect different components of  $\partial\Omega$  and lie on the zero level of the energy. We allow that  $\partial\Omega$  contains a finite number of critical points of  $U$ . The case of symmetric potential is also considered.

## 1 Introduction

Let  $U : \mathbb{R}^n \rightarrow \mathbb{R}$  be a function of class  $C^2$ . We assume that  $\Omega \subset \mathbb{R}^n$  is a connected component of the set  $\{x \in \mathbb{R}^n : U(x) > 0\}$  and that  $\partial\Omega$  is compact and is the union of  $N \geq 1$  distinct nonempty connected components  $\Gamma_1, \dots, \Gamma_N$ . We consider the following situations

**H**  $N \geq 2$  and, if  $\Omega$  is unbounded, there is  $r_0 > 0$  and a non-negative function  $\sigma : [r_0, +\infty) \rightarrow \mathbb{R}$  such that  $\int_{r_0}^{+\infty} \sigma(r) dr = +\infty$  and

$$\sqrt{U(x)} \geq \sigma(|x|), \quad x \in \Omega, \quad |x| \geq r_0. \quad (1.1)$$

**H<sub>s</sub>**  $\Omega$  is bounded, the origin  $0 \in \mathbb{R}^n$  belongs to  $\Omega$  and  $U$  is invariant under the antipodal map

$$U(-x) = U(x), \quad x \in \Omega.$$

Condition (1.1) was first introduced in [7]. A sufficient condition for (1.1) is that  $\liminf_{|x| \rightarrow \infty} U(x) > 0$ .

We study non constant solutions  $u : (T_-, T_+) \rightarrow \Omega$ , of the equation

$$\ddot{u} = U_x(u), \quad U_x = \left( \frac{\partial U}{\partial x} \right)^T, \quad (1.2)$$

that satisfy

$$\lim_{t \rightarrow T_{\pm}} d(u(t), \partial\Omega) = 0, \quad (1.3)$$

with  $d$  the Euclidean distance, and lie on the energy surface

$$\frac{1}{2} |\dot{u}|^2 - U(u) = 0. \quad (1.4)$$

We allow that the boundary  $\partial\Omega$  of  $\Omega$  contains a finite set  $P$  of critical points of  $U$  and assume

<sup>\*</sup>Dipartimento di Matematica, Università dell'Aquila; e-mail: [fusco@univaq.it](mailto:fusco@univaq.it)

<sup>†</sup>Dipartimento di Matematica, Università di Pisa; e-mail: [giovanni.federico.gronchi@unipi.it](mailto:giovanni.federico.gronchi@unipi.it)

<sup>‡</sup>Dipartimento di Matematica, Università di Pisa; e-mail: [matteo.novaga@unipi.it](mailto:matteo.novaga@unipi.it)

**H<sub>1</sub>** If  $\Gamma \in \{\Gamma_1, \dots, \Gamma_N\}$  has positive diameter and  $p \in P \cap \Gamma$  then  $p$  is a hyperbolic critical point of  $U$ .

If  $\Gamma$  has positive diameter, then hyperbolic critical points  $p \in \Gamma$  correspond to saddle-center equilibrium points in the zero energy level of the Hamiltonian system associated to (1.2). These points are organizing centers of complex dynamics, see [6].

Note that **H<sub>1</sub>** does not exclude that some of the  $\Gamma_j$  reduce to a singleton, say  $\{p\}$ , for some  $p \in P$ . In this case nothing is required on the behavior of  $U$  in a neighborhood of  $p$  aside from being  $C^2$ .

A comment on **H** and **H<sub>s</sub>** is in order. If  $P$  is nonempty  $u \equiv p$  for  $p \in P$  is a constant solution of (1.2) that satisfies (1.3) and (1.4). To avoid trivial solutions of this kind we require  $N \geq 2$  in **H**, and look for solutions that connect different components of  $\partial\Omega$ . In **H<sub>s</sub>** we do not exclude that  $\partial\Omega$  is connected ( $N = 1$ ) and avoid trivial solutions by restricting to a symmetric context and to solutions that pass through 0.

We prove the following results.

**Theorem 1.1.** *Assume that **H** and **H<sub>1</sub>** hold. Then for each  $\Gamma_- \in \{\Gamma_1, \dots, \Gamma_N\}$  there exist  $\Gamma_+ \in \{\Gamma_1, \dots, \Gamma_N\} \setminus \{\Gamma_-\}$  and a map  $u^* : (T_-, T_+) \rightarrow \Omega$ , with  $-\infty \leq T_- < T_+ \leq +\infty$ , that satisfies (1.2), (1.4) and*

$$\lim_{t \rightarrow T_{\pm}} d(u^*(t), \Gamma_{\pm}) = 0. \quad (1.5)$$

Moreover,  $T_- > -\infty$  (resp.  $T_+ < +\infty$ ) if and only if  $\Gamma_-$  (resp.  $\Gamma_+$ ) has positive diameter. If  $T_- > -\infty$  it results

$$\begin{aligned} \lim_{t \rightarrow T_-} u^*(t) &= x_-, \\ \lim_{t \rightarrow T_-} \dot{u}^*(t) &= 0, \end{aligned} \quad (1.6)$$

for some  $x_- \in \Gamma_- \setminus P$ . An analogous statement holds if  $T_+ < +\infty$ .

**Theorem 1.2.** *Assume that **H<sub>s</sub>** and **H<sub>1</sub>** hold. Then there exist  $\Gamma_+ \in \{\Gamma_1, \dots, \Gamma_N\}$  and a map  $u^* : (0, T_+) \rightarrow \Omega$ , with  $0 < T_+ \leq +\infty$ , that satisfies (1.2), (1.4) and*

$$\lim_{t \rightarrow T_+} d(u^*(t), \Gamma_+) = 0.$$

Moreover,  $T_+ < +\infty$  if and only if  $\Gamma_+$  has positive diameter. If  $T_+ < +\infty$  it results

$$\begin{aligned} \lim_{t \rightarrow T_+} u^*(t) &= x_+, \\ \lim_{t \rightarrow T_+} \dot{u}^*(t) &= 0, \end{aligned}$$

for some  $x_+ \in \Gamma_+ \setminus P$ .

We list a few straightforward consequences of Theorems 1.1 and 1.2.

**Corollary 1.3.** *Theorem 1.1 implies that, if  $\partial\Omega = P$ , given  $p_- \in P$  there is  $p_+ \in P \setminus \{p_-\}$  and a heteroclinic connection between  $p_-$  and  $p_+$ , that is a solution  $u^* : \mathbb{R} \rightarrow \mathbb{R}^n$  of (1.2) and (1.4) that satisfies*

$$\lim_{t \rightarrow \pm\infty} u^*(t) = p_{\pm}.$$

The problem of the existence of heteroclinic connections between two isolated zeros  $p_{\pm}$  of a non-negative potential has been recently reconsidered by several authors. In [1] existence was established under a mild monotonicity condition on  $U$  near  $p_{\pm}$ . This condition was removed in [8], see also [2]. The most general results, equivalent to the consequence of Theorem 1.1 discussed in Section 2.1, were recently obtained in [7] and in [11], see also [3]. All these papers establish existence by a variational approach. In [1], [8] and [2] by minimizing the action functional, and in [7] and [11] by minimizing the Jacobi functional.

**Corollary 1.4.** *Theorem 1.1 implies that, if  $\Gamma_- = \{p\}$  for some  $p \in P$  and the elements of  $\{\Gamma_1, \dots, \Gamma_N\} \setminus \{\Gamma_-\}$  have all positive diameter, there exists a nontrivial orbit homoclinic to  $p$  that satisfies (1.2), (1.4).*

*Proof.* Let  $v^* : \mathbb{R} \rightarrow \Omega \cup \{x_+\}$  be the extension defined by

$$v^*(T_+ + t) = u^*(T_+ - t), \quad t \in (0, +\infty), \quad v^*(T_+) = x_+,$$

of the solution  $u^* : (-\infty, T_+) \rightarrow \Omega$  given by Theorem 1.1. The map  $v^*$  so defined is a smooth non-constant solution of (1.2) that satisfies

$$\lim_{t \rightarrow \pm\infty} v^*(t) = p.$$

□

**Corollary 1.5.** *Theorem 1.1 implies that, if all the sets  $\Gamma_1, \dots, \Gamma_N$  have positive diameter, given  $\Gamma_- \in \{\Gamma_1, \dots, \Gamma_N\}$ , there exist  $\Gamma_+ \in \{\Gamma_1, \dots, \Gamma_N\} \setminus \{\Gamma_-\}$  and a periodic solution  $v^* : \mathbb{R} \rightarrow \Omega$  of (1.2) and (1.4) that oscillates between  $\Gamma_-$  and  $\Gamma_+$ . This solution has period  $T = 2(T_+ - T_-)$ .*

*Proof.* The solution  $v^*$  is the  $T$ -periodic extension of the map  $w^* : [T_-, 2T_+ - T_-] \rightarrow \Omega$  defined by  $w^*(t) = u^*(t)$  for  $t \in (T_-, T_+)$ , where  $u^*$  is given by Theorem 1.1, and

$$\begin{aligned} w^*(T_\pm) &= x_\pm, \\ w^*(T_+ + t) &= u^*(T_+ - t), \quad t \in (0, T_+ - T_-]. \end{aligned}$$

□

The problem of existence of heteroclinic, homoclinic and periodic solutions of (1.2), in a context similar to the one considered here, was already discussed in [2] where  $\partial\Omega$  is allowed to include continua of critical points. Our result concerning periodic solutions extends a corresponding result in [2] where existence was established under the assumption that  $P = \emptyset$ .

The following result is a direct consequence of Theorem 1.2.

**Corollary 1.6.** *Theorem 1.2 implies that, if all the sets  $\Gamma_1, \dots, \Gamma_N$  have positive diameter, there exists  $\Gamma_+ \in \{\Gamma_1, \dots, \Gamma_N\}$  and a periodic solution  $v^* : \mathbb{R} \rightarrow \Omega$  of (1.2) and (1.4) that satisfies*

$$v^*(-t) = -v^*(t), \quad t \in \mathbb{R}.$$

*This solution has period  $T = 4T_+$ , with  $T_+$ .*

*Proof.* The solution  $v^*$  is the  $T$ -periodic extension of the map  $w^* : [-2T_+, 2T_+] \rightarrow \Omega$  defined by  $w^*(t) = u^*(t)$  for  $t \in (0, T_+)$ , where  $u^*$  is given by Theorem 1.2, and by

$$\begin{aligned} w^*(t) &= -w^*(-t), & t \in (-T_+, 0), \\ w^*(0) &= 0, \quad w^*(\pm T_+) = \pm x_+, \\ w^*(T_+ + t) &= w^*(T_+ - t), & t \in (0, T_+], \\ w^*(-T_+ + t) &= w^*(-T_+ - t), & t \in [-T_+, 0). \end{aligned}$$

In particular the solution oscillates between  $x_+$  and  $-x_+$  and this is true also when  $\partial\Omega$  is connected ( $N = 1$ ). □

## 2 Proof of Theorems 1.1 and 1.2

We recall a classical result.

**Lemma 2.1.** *Let  $G : \mathbb{R}^n \rightarrow \mathbb{R}$  be a smooth bounded and non-negative potential,  $I = (a, b)$  a bounded interval. Define the Jacobi functional*

$$\mathcal{J}_G(q, I) = \sqrt{2} \int_I \sqrt{G(q(t))} |\dot{q}(t)| dt$$

and the action functional

$$\mathcal{A}_G(q, I) = \int_I \left( \frac{1}{2} |\dot{q}(t)|^2 + G(q(t)) \right) dt.$$

Then

(i)

$$\mathcal{J}_G(q, I) \leq \mathcal{A}_G(q, I), \quad q \in W^{1,2}(I; \mathbb{R}^n)$$

with equality sign if and only if

$$\frac{1}{2} |\dot{q}(t)|^2 - G(q(t)) = 0, \quad t \in I.$$

(ii)

$$\min_{q \in \mathcal{Q}} \mathcal{J}_G(q, I) = \min_{q \in \mathcal{Q}} \mathcal{A}_G(q, I),$$

where

$$\mathcal{Q} = \{q \in W^{1,2}(I; \mathbb{R}^n) : q(a) = q_a, q(b) = q_b\}.$$

When  $G = U$  we shall simply write  $\mathcal{J}, \mathcal{A}$  for  $\mathcal{J}_U, \mathcal{A}_U$ .

We now start the proof of Theorem 1.1. Choose  $\Gamma_- \in \{\Gamma_1, \dots, \Gamma_N\}$  and set

$$d = \min\{|x - y| : x \in \Gamma_-, y \in \partial\Omega \setminus \Gamma_-\}.$$

For small  $\delta \in (0, d)$  let  $O_\delta = \{x \in \Omega : d(x, \Gamma_-) < \delta\}$  and let  $U_0 = \frac{1}{2} \min_{x \in \partial O_\delta \cap \Omega} U(x)$ . We note that  $U_0 > 0$  and define the admissible set

$$\begin{aligned} \mathcal{U} = \{ & u \in W^{1,2}((T_-^u, T_+^u); \mathbb{R}^n) : -\infty < T_-^u < T_+^u < +\infty, \\ & u((T_-^u, T_+^u)) \subset \Omega, U(u(0)) = U_0, u(T_-^u) \in \Gamma_-, u(T_+^u) \in \partial\Omega \setminus \Gamma_- \}. \end{aligned} \quad (2.1)$$

We determine the map  $u^*$  in Theorem 1.1 as the limit of a minimizing sequence  $\{u_j\} \subset \mathcal{U}$  of the action functional

$$\mathcal{A}(u, (T_-^u, T_+^u)) = \int_{T_-^u}^{T_+^u} \left( \frac{1}{2} |\dot{u}(t)|^2 + U(u(t)) \right) dt,$$

Note that in the definition of  $\mathcal{U}$  the times  $T_-^u$  and  $T_+^u$  are not fixed but, in general, change with  $u$ . Note also that the condition  $U(u(0)) = U_0$  in (2.1) is a normalization which can always be imposed by a translation of time and has the scope of eliminating the loss of compactness due to translation invariance. Let  $\bar{x}_- \in \Gamma_-$  and  $\bar{x}_+ \in \partial\Omega \setminus \Gamma_-$  be such that  $|\bar{x}_+ - \bar{x}_-| = d$  and set

$$\tilde{u}(t) = (1 - (t + \tau))\bar{x}_- + (t + \tau)\bar{x}_+, \quad t \in [-\tau, 1 - \tau],$$

where  $\tau \in (0, 1)$  is chosen so that  $U(\tilde{u}(0)) = U_0$ . Then  $\tilde{u} \in \mathcal{U}$ ,  $T_-^{\tilde{u}} = -\tau$ ,  $T_+^{\tilde{u}} = 1 - \tau$  and

$$\mathcal{A}(\tilde{u}, (-\tau, 1 - \tau)) = a < +\infty.$$

Next we show that there are constants  $M > 0$  and  $T_0 > 0$  such that each  $u \in \mathcal{U}$  with

$$\mathcal{A}(u, (T_-^u, T_+^u)) \leq a, \quad (2.2)$$

satisfies

$$\begin{aligned} \|u\|_{L^\infty((T_-^u, T_+^u); \mathbb{R}^n)} &\leq M, \\ T_-^u &\leq -T_0 < T_0 \leq T_+^u. \end{aligned} \quad (2.3)$$

The  $L^\infty$  bound on  $u$  follows from **H** and from Lemma 2.1, in fact, if  $\Omega$  is unbounded,  $|u(\bar{t})| = M$  for some  $\bar{t} \in (T_-^u, T_+^u)$  implies

$$a \geq \mathcal{A}(u, (T_-^u, \bar{t})) \geq \int_{T_-^u}^{\bar{t}} \sqrt{2U(u(t))} |\dot{u}(t)| dt \geq \sqrt{2} \int_{r_0}^M \sigma(s) ds.$$

The existence of  $T_0$  follows from

$$\frac{d_1^2}{|T_-^u|} \leq \int_{T_-^u}^0 |\dot{u}(t)|^2 dt \leq 2a, \quad \frac{d_1^2}{T_+^u} \leq \int_0^{T_+^u} |\dot{u}(t)|^2 dt \leq 2a,$$

where  $d_1 = d(\partial\Omega, \{x : U(x) > U_0\})$ .

Let  $\{u_j\} \subset \mathcal{U}$  be a minimizing sequence

$$\lim_{j \rightarrow +\infty} \mathcal{A}(u_j, (T_-^{u_j}, T_+^{u_j})) = \inf_{u \in \mathcal{U}} \mathcal{A}(u, (T_-^u, T_+^u)) := a_0 \leq a. \quad (2.4)$$

We can assume that each  $u_j$  satisfies (2.2) and (2.3). By considering a subsequence, that we still denote by  $\{u_j\}$ , we can also assume that there exist  $T_-^\infty, T_+^\infty$  with  $-\infty \leq T_-^\infty \leq -T_0 < T_0 \leq T_+^\infty \leq +\infty$  and a continuous map  $u^* : (T_-^\infty, T_+^\infty) \rightarrow \mathbb{R}^n$  such that

$$\begin{aligned} \lim_{j \rightarrow +\infty} T_\pm^{u_j} &= T_\pm^\infty, \\ \lim_{j \rightarrow +\infty} u_j(t) &= u^*(t), \quad t \in (T_-^\infty, T_+^\infty), \end{aligned} \quad (2.5)$$

and in the last limit the convergence is uniform on bounded intervals. This follows from (2.3) which implies that the sequence  $\{u_j\}$  is equi-bounded and from (2.2) which implies

$$|u_j(t_1) - u_j(t_2)| \leq \left| \int_{t_1}^{t_2} |\dot{u}_j(t)| dt \right| \leq \sqrt{a} |t_1 - t_2|^{\frac{1}{2}}, \quad (2.6)$$

so that the sequence is also equi-continuous.

By passing to a further subsequence we can also assume that  $u_j \rightharpoonup u^*$  in  $W^{1,2}((T_1, T_2); \mathbb{R}^n)$  for each  $T_1, T_2$  with  $T_-^\infty < T_1 < T_2 < T_+^\infty$ . This follows from (2.2), which implies

$$\frac{1}{2} \int_{T_-^{u_j}}^{T_+^{u_j}} |\dot{u}_j|^2 dt \leq \mathcal{A}(u_j, (T_-^{u_j}, T_+^{u_j})) \leq a,$$

and from the fact that each map  $u_j$  satisfies (2.3) and therefore is bounded in  $L^2((T_-^{u_j}, T_+^{u_j}); \mathbb{R}^n)$ .

We also have

$$\mathcal{A}(u^*, (T_-^\infty, T_+^\infty)) \leq a_0. \quad (2.7)$$

Indeed, from the lower semicontinuity of the norm, for each  $T_1, T_2$  with  $T_-^\infty < T_1 < T_2 < T_+^\infty$  we have

$$\int_{T_1}^{T_2} |\dot{u}^*|^2 dt \leq \liminf_{j \rightarrow +\infty} \int_{T_1}^{T_2} |\dot{u}_j|^2 dt.$$

This and the fact that  $u_j$  converges to  $u^*$  uniformly in  $[T_1, T_2]$  imply

$$\mathcal{A}(u^*, (T_1, T_2)) \leq \liminf_{j \rightarrow +\infty} \mathcal{A}(u_j, (T_1, T_2)) \leq \liminf_{j \rightarrow +\infty} \mathcal{A}(u_j, (T_-^{u_j}, T_+^{u_j})) = a_0.$$

Since this is valid for each  $T_-^\infty < T_1 < T_2 < T_+^\infty$  the claim (2.7) follows.

**Lemma 2.2.** Define  $T_-^\infty \leq T_- \leq -T_0 < T_0 \leq T_+ \leq T_+^\infty$  by setting

$$\begin{aligned} T_- &= \inf\{t \in (T_-^\infty, 0] : u^*((t, 0]) \subset \Omega\} \\ T_+ &= \sup\{t \in (0, T_+^\infty) : u^*([0, t]) \subset \Omega\}. \end{aligned}$$

Then

$$(i) \quad \mathcal{A}(u^*, (T_-, T_+)) = a_0. \quad (2.8)$$

(ii)  $T_+ < +\infty$  implies  $\lim_{t \rightarrow T_+} u^*(t) = x_+$  for some  $x_+ \in \Gamma_+$  and  $\Gamma_+ \in \{\Gamma_1, \dots, \Gamma_N\} \setminus \{\Gamma_-\}$ .

(iii)  $T_+ = +\infty$  implies

$$\lim_{t \rightarrow +\infty} d(u^*(t), \Gamma_+) = 0, \quad (2.9)$$

for some  $\Gamma_+ \in \{\Gamma_1, \dots, \Gamma_N\} \setminus \{\Gamma_-\}$ .

Corresponding statements apply to  $T_-$ .

*Proof.* We first prove (ii), (iii). If  $T_+ < +\infty$  the existence of  $\lim_{t \rightarrow T_+} u^*(t)$  follows from (2.6) which implies that  $u^*$  is a  $C^{0, \frac{1}{2}}$  map. The limit  $x_+$  belongs to  $\partial\Omega$  and therefore to  $\Gamma_+$  for some  $\Gamma_+ \in \{\Gamma_1, \dots, \Gamma_N\}$ . Indeed,  $x_+ \notin \partial\Omega$  would imply the existence of  $\tau > 0$  such that, for  $j$  large enough,

$$d(u_j([T_+, T_+ + \tau]), \partial\Omega) \geq \frac{1}{2}d(x_+, \partial\Omega),$$

in contradiction with the definition of  $T_+$ . If  $T_+ = +\infty$  and (iii) does not hold there is  $\delta > 0$  and a diverging sequence  $\{t_j\}$  such that

$$d(u^*(t_j), \partial\Omega) \geq \delta.$$

Set  $U_m = \min_{d(x, \partial\Omega) = \delta} U(x) > 0$ . From the uniform continuity of  $U$  in  $\{|x| \leq M\}$  ( $M$  as in (2.3)) it follows that there is  $l > 0$  such that

$$|U(x_1) - U(x_2)| \leq \frac{1}{2}U_m, \quad \text{for } |x_1 - x_2| \leq l, \quad x_1, x_2 \in \{|x| \leq M\}.$$

This and  $u^* \in C^{0, \frac{1}{2}}$  imply

$$U(u^*(t)) \geq \frac{1}{2}U_m, \quad t \in I_j = \left(t_j - \frac{l^2}{a}, t_j + \frac{l^2}{a}\right),$$

and, by passing to a subsequence, we can assume that the intervals  $I_j$  are disjoint. Therefore for each  $T > 0$  we have

$$\sum_{t_j \leq T} \frac{l^2 U_m}{a} \leq \int_0^T U(u^*(t)) dt \leq a_0,$$

which is impossible for  $T$  large. This establishes (2.9) for some  $\Gamma_+ \in \{\Gamma_1, \dots, \Gamma_N\}$ . It remains to show that  $\Gamma_+ \neq \Gamma_-$ . This is a consequence of the minimizing character of  $\{u_j\}$ . Indeed,  $\Gamma_+ = \Gamma_-$  would imply the existence of a constant  $c > 0$  such that  $\lim_{j \rightarrow \infty} \mathcal{A}(u_j, (T_-^{u_j}, T_+^{u_j})) \geq a_0 + c$ .

Now we prove (i).  $T_+ - T_- < +\infty$ , implies that  $u^*$  is an element of  $\mathcal{U}$  with  $T_{\pm}^{u^*} = T_{\pm}$ . It follows that  $\mathcal{A}(u^*, (T_-, T_+)) \geq a_0$ , which together with (2.7) imply (2.8). Assume now  $T_+ - T_- = +\infty$ . If  $T_+ = +\infty$ , (2.9) implies that, given a small number  $\epsilon > 0$ , there are  $t_\epsilon$  and  $\bar{x}_\epsilon \in \partial\Omega$  such that  $|u^*(t_\epsilon) - \bar{x}_\epsilon| = \epsilon$  and the segment joining  $u^*(t_\epsilon)$  to  $\bar{x}_\epsilon$  belongs to  $\bar{\Omega}$ . Set

$$v_\epsilon(t) = (1 - (t - t_\epsilon))u^*(t_\epsilon) + (t - t_\epsilon)\bar{x}_\epsilon, \quad t \in (t_\epsilon, t_\epsilon + 1].$$

From the uniform continuity of  $U$  there is  $\eta_\epsilon > 0$ ,  $\lim_{\epsilon \rightarrow 0} \eta_\epsilon = 0$ , such that  $U(v_\epsilon(t)) \leq \eta_\epsilon$ , for  $t \in [t_\epsilon, t_\epsilon + 1]$ . Therefore we have

$$\mathcal{A}(v_\epsilon, (t_\epsilon, t_\epsilon + 1)) \leq \frac{1}{2}\epsilon^2 + \eta_\epsilon.$$

If  $T_- > -\infty$  the map  $u_\epsilon = \mathbb{1}_{[T_-, t_\epsilon]}u^* + \mathbb{1}_{(t_\epsilon, t_\epsilon + 1]}v_\epsilon$  belongs to  $\mathcal{U}$  and it results

$$a_0 \leq \mathcal{A}(u_\epsilon, (T_-, t_\epsilon + 1)) = \mathcal{A}(u^*, (T_-, t_\epsilon)) + \mathcal{A}(v_\epsilon, (t_\epsilon, t_\epsilon + 1)) \leq \mathcal{A}(u^*, (T_-, T_+)) + \frac{1}{2}\epsilon^2 + \eta_\epsilon.$$

Since this is valid for all small  $\epsilon > 0$  we get

$$a_0 \leq \mathcal{A}(u^*, (T_-, T_+)),$$

that together with (2.7) establishes (2.8) if  $T_- > -\infty$  and  $T_+ = +\infty$ . The discussion of the other cases where  $T_+ - T_- = +\infty$  is similar.  $\square$

We observe that there are cases with  $T_+ < T_+^\infty$  and/or  $T_- > T_-^\infty$ , see Remark 2.

**Lemma 2.3.** *The map  $u^*$  satisfies (1.2) and (1.4) in  $(T_-, T_+)$ .*

*Proof.* 1. We first show that for each  $T_1, T_2$  with  $T_- < T_1 < T_2 < T_+$  we have

$$\mathcal{A}(u^*, (T_1, T_2)) = \inf_{v \in \mathcal{V}} \mathcal{A}(v, (T_1, T_2)), \quad (2.10)$$

where

$$\mathcal{V} = \{v \in W^{1,2}((T_1, T_2); \mathbb{R}^n) : v(T_i) = u^*(T_i), i = 1, 2; v([T_1, T_2]) \subset \Omega\}.$$

Suppose instead that there are  $\eta > 0$  and  $v \in \mathcal{V}$  such that

$$\mathcal{A}(v, (T_1, T_2)) = \mathcal{A}(u^*, (T_1, T_2)) - \eta.$$

Set  $w_j : (T_-^{u_j}, T_+^{u_j}) \rightarrow \Omega$  defined by

$$w_j(t) = \begin{cases} u_j(t), & t \in (T_-^{u_j}, T_1] \cup [T_2, T_+^{u_j}), \\ v(t) + \frac{T_2 - t}{T_2 - T_1}\delta_{1j} + \frac{t - T_1}{T_2 - T_1}\delta_{2j}, & t \in (T_1, T_2), \end{cases}$$

where  $\delta_{ij} = u_j(T_i) - u^*(T_i)$ ,  $i = 1, 2$ , with  $u_j$  as in (2.4). Define  $v_j : [T_-^{v_j}, T_+^{v_j}] \rightarrow \mathbb{R}^n$  by

$$v_j(t) = w_j(t - \tau_j),$$

where  $\tau_j$  is such that  $U(v_j(0)) = U_0$ , as in (2.1). Note that

$$\mathcal{A}(v_j, (T_-^{v_j}, T_+^{v_j})) = \mathcal{A}(w_j, (T_-^{u_j}, T_+^{u_j})). \quad (2.11)$$

From (2.5) we have  $\lim_{j \rightarrow \infty} \delta_{ij} = 0$ ,  $i = 1, 2$ , so that

$$\lim_{j \rightarrow +\infty} \mathcal{A}(w_j, (T_1, T_2)) = \mathcal{A}(v, (T_1, T_2)) = \mathcal{A}(u^*, (T_1, T_2)) - \eta \leq \liminf_{j \rightarrow +\infty} \mathcal{A}(u_j, (T_1, T_2)) - \eta.$$

Therefore we have

$$\begin{aligned} \liminf_{j \rightarrow +\infty} \mathcal{A}(w_j, (T_-^{u_j}, T_+^{u_j})) &= \lim_{j \rightarrow +\infty} \mathcal{A}(w_j, (T_1, T_2)) + \liminf_{j \rightarrow +\infty} \mathcal{A}(u_j, (T_+^{u_j}, T_1) \cup (T_2, T_+^{u_j})) \\ &\leq \liminf_{j \rightarrow +\infty} \mathcal{A}(u_j, (T_1, T_2)) - \eta + \liminf_{j \rightarrow +\infty} \mathcal{A}(u_j, (T_+^{u_j}, T_1) \cup (T_2, T_+^{u_j})) \leq a_0 - \eta, \end{aligned}$$

that, given (2.11), is in contradiction with the minimizing character of the sequence  $\{u_j\}$ .

The fact that  $u^*$  satisfies (1.2) follows from (2.10) and regularity theory, see [5]. To show that  $u^*$  satisfies (1.4) we distinguish the case  $T_+ - T_- < +\infty$  from the case  $T_+ - T_- = +\infty$ .

2.  $T_+ - T_- < +\infty$ . Given  $t_0, t_1$  with  $T_- < t_0 < t_1 < T_+$ , let  $\phi : [t_0, t_1 + \tau] \rightarrow [t_0, t_1]$  be linear, with  $|\tau|$  small, and let  $\psi : [t_0, t_1] \rightarrow [t_0, t_1 + \tau]$  be the inverse of  $\phi$ . Define  $u_\tau : [T_-, T_+ + \tau] \rightarrow \mathbb{R}^n$  by setting

$$u_\tau(t) = \begin{cases} u^*(t), & t \in [T_-, t_0], \\ u^*(\phi(t)), & t \in [t_0, t_1 + \tau], \\ u^*(t - \tau), & t \in (t_1 + \tau, T_+ + \tau) \end{cases} \quad (2.12)$$

Note that  $u_\tau \in \mathcal{U}$  with  $T_-^{u_\tau} = T_-$  and  $T_+^{u_\tau} = T_+ + \tau$ . Since  $u^*$  is a minimizer we have

$$\frac{d}{d\tau} \mathcal{A}(u_\tau, (T_-^{u_\tau}, T_+^{u_\tau}))|_{\tau=0} = 0. \quad (2.13)$$

From (2.12), using also the change of variables  $t = \psi(s)$ , it follows

$$\begin{aligned} &\mathcal{A}(u_\tau, (T_-^{u_\tau}, T_+^{u_\tau})) - \mathcal{A}(u^*, (T_-, T_+)) \\ &= \int_{t_0}^{t_1 + \tau} \left( \frac{\dot{\phi}^2(t)}{2} |\dot{u}^*(\phi(t))|^2 + U(u^*(\phi(t))) \right) dt - \int_{t_0}^{t_1} \left( \frac{1}{2} |\dot{u}^*(t)|^2 + U(u^*(t)) \right) dt \\ &= \int_{t_0}^{t_1} \left( \frac{1 - \dot{\psi}(t)}{2\dot{\psi}(t)} |\dot{u}^*(t)|^2 + (\dot{\psi}(t) - 1)U(u^*(t)) \right) dt \\ &= \int_{t_0}^{t_1} \left( \frac{-\frac{\tau}{t_1 - t_0}}{2(1 + \frac{\tau}{t_1 - t_0})} |\dot{u}^*(t)|^2 + \frac{\tau}{t_1 - t_0} U(u^*(t)) \right) dt \\ &= -\frac{\tau}{t_1 - t_0} \int_{t_0}^{t_1} \left( \frac{|\dot{u}^*(t)|^2}{2(1 + \frac{\tau}{t_1 - t_0})} - U(u^*(t)) \right) dt. \end{aligned}$$

This and (2.13) imply

$$\int_{t_0}^{t_1} \left( \frac{1}{2} |\dot{u}^*(t)|^2 - U(u^*(t)) \right) dt = 0. \quad (2.14)$$

Since this holds for all  $t_0, t_1$ , with  $T_- < t_0 < t_1 < T_+$ , then (1.4) follows.

3.  $T_+ - T_- = +\infty$ . We only consider the case  $T_+ = +\infty$ . The discussion of the other cases is similar. Let  $T \in (T_-, +\infty)$ , let  $T_- < t_0 < t_1 < T$  and let  $\phi : [t_0, T] \rightarrow [t_0, T]$  be linear in the intervals  $[t_0, t_1 + \tau]$ ,  $[t_1 + \tau, T]$ , with  $|\tau|$  small, and such that  $\phi([t_0, t_1 + \tau]) = [t_0, t_1]$ . Define  $u_\tau : (T_-, +\infty) \rightarrow \mathbb{R}^n$  by setting

$$u_\tau(t) = \begin{cases} u^*(t), & t \in (T_-, t_0] \cup [T, +\infty) \\ u^*(\phi(t)), & t \in [t_0, T]. \end{cases}$$

We have

$$\begin{aligned} &\mathcal{A}(u_\tau, (T_-, T)) - \mathcal{A}(u^*, (T_-, T)) \\ &= \int_{t_0}^{t_1} \left( \frac{-\frac{\tau}{t_1 - t_0}}{2(1 + \frac{\tau}{t_1 - t_0})} |\dot{u}^*(t)|^2 + \frac{\tau}{t_1 - t_0} U(u^*(t)) \right) dt + \int_{t_1}^T \left( \frac{\frac{\tau}{T - t_1}}{2(1 + \frac{\tau}{T - t_1})} |\dot{u}^*(t)|^2 - \frac{\tau}{T - t_1} U(u^*(t)) \right) dt. \end{aligned}$$



Since  $u^*$  restricted to the interval  $[t_0, T]$  is a minimizer of (2.10), by differentiating with respect to  $\tau$  and setting  $\tau = 0$  we obtain

$$-\frac{1}{t_1 - t_0} \int_{t_0}^{t_1} \left( \frac{1}{2} |\dot{u}^*(t)|^2 - U(u^*(t)) \right) dt + \frac{1}{T - t_1} \int_{t_1}^T \left( \frac{1}{2} |\dot{u}^*(t)|^2 - U(u^*(t)) \right) dt = 0.$$

From (2.7) it follows that the second term in this expression converges to zero when  $T \rightarrow +\infty$ . Therefore, after taking the limit for  $T \rightarrow +\infty$ , we get back to (2.14) and, as before, we conclude that (1.4) holds.  $\square$

**Lemma 2.4.** *Assume that  $\lim_{t \rightarrow T_+} u^*(t) = p \in P$ . Then*

$$T_+ = +\infty.$$

*Proof.* Since  $U$  is of class  $C^2$  and  $p$  is a critical point of  $U$  there are constants  $c > 0$  and  $\rho > 0$  such that

$$U(x) \leq c|x - p|^2, \quad x \in B_\rho(p) \cap \Omega.$$

Fix  $t_\rho$  so that  $u^*(t) \in B_\rho(p) \cap \Omega$  for  $t \geq t_\rho$ . Then  $T_+ = +\infty$  follows from (1.4) and

$$\frac{d}{dt} |u^* - p| \geq -|\dot{u}^*| = -\sqrt{2U(u^*)} \geq -\sqrt{2c}|u^* - p|, \quad t \geq t_\rho.$$

$\square$

We now show that if  $\Gamma_+$  has positive diameter then  $T_+ < +\infty$ . To prove this we first show that  $T_+ = +\infty$  implies  $u^*(t) \rightarrow p \in P$  as  $t \rightarrow +\infty$ , then we conclude that this is in contrast with (2.8).

**Lemma 2.5.** *If  $T_+ = +\infty$ , then there is  $p \in P$  such that*

$$\lim_{t \rightarrow +\infty} u^*(t) = p. \quad (2.15)$$

*An analogous statement applies to  $T_-$ .*

*Proof.* If  $\Gamma_+ = \{p\}$  for some  $p \in P$ , then (2.15) follows by (2.9). Therefore we assume that  $\Gamma_+$  has positive diameter. The idea of the proof is to show that if  $u^*(t)$  gets too close to  $\partial\Gamma_+ \setminus P$  it is forced to end up on  $\Gamma_+ \setminus P$  in a finite time in contradiction with  $T^* = +\infty$ .

If (2.15) does not hold there is  $q > 0$  and a sequence  $\{\tau_j\}$ , with  $\lim_{j \rightarrow \infty} \tau_j = +\infty$ , such that  $d(u^*(\tau_j), P) \geq q$ , for all  $j \in \mathbb{N}$ . Since, by (2.3)  $u^*$  is bounded, using also (2.9), we can assume that

$$\lim_{j \rightarrow +\infty} u^*(\tau_j) = \bar{x}, \quad \text{for some } \bar{x} \in \Gamma_+ \setminus \cup_{p \in P} B_q(p). \quad (2.16)$$

The smoothness of  $U$  implies that there are positive constants  $\bar{r}$ ,  $r$ ,  $c$  and  $C$  such that

(i) the orthogonal projection on  $\pi : B_{\bar{r}}(\bar{x}) \rightarrow \partial\Omega$  is well defined and  $\pi(B_{\bar{r}}(\bar{x})) \subset \partial\Omega \setminus P$ ;

(ii) we have

$$B_r(x_0) \subset B_{\bar{r}}(\bar{x}), \quad \text{for all } x_0 \in \partial\Omega \cap B_{\frac{\bar{r}}{2}}(\bar{x});$$

(iii) if  $(\xi, s) \in \mathbb{R}^{n-1} \times \mathbb{R}$  are local coordinates with respect to a basis  $\{e_1, \dots, e_n\}$ ,  $e_j = e_j(x_0)$ , with  $e_n(x_0)$  the unit interior normal to  $\partial\Omega$  at  $x_0 \in \partial\Omega \cap B_{\frac{\bar{r}}{2}}(\bar{x})$  it results

$$\frac{1}{2}cs \leq U(x(x_0, (\xi, s))) \leq 2cs, \quad |\xi|^2 + s^2 \leq r^2, \quad s \geq h(x_0, \xi), \quad (2.17)$$

where

$$x = x(x_0, (\xi, s)) = x_0 + \sum_{j=1}^n \xi_j e_j(x_0) + s e_n(x_0),$$

and  $h : \partial\Omega \cap B_{\frac{\bar{r}}{2}}(\bar{x}) \times \{|\xi| \leq r\} \rightarrow \mathbb{R}$ ,  $|h(x_0, \xi)| \leq C|\xi|^2$ , for  $|\xi| \leq r$ , is a local representation of  $\partial\Omega$  in a neighborhood of  $x_0$ , that is  $U(x(x_0, (\xi, h(x_0, \xi)))) = 0$  for  $|\xi| \leq r$ .

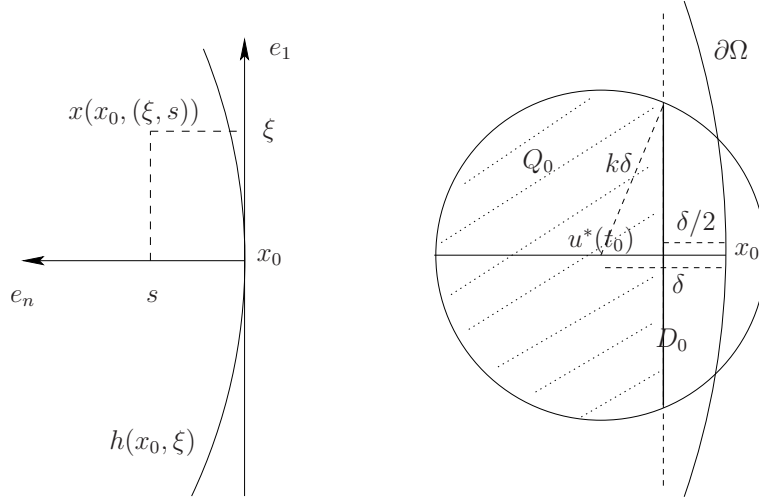


Figure 1: The coordinates  $(\xi, s)$  and the domain  $Q_0$  in Lemma 2.5.

Fix a value  $j_0$  of  $j$  and set  $t_0 = \tau_{j_0}$ . If  $j_0$  is sufficiently large, setting  $t_0 = \tau_{j_0}$  we have that  $x_0 = \pi(u^*(t_0))$  is well defined. Moreover  $x_0 \in \partial\Omega \cap B_{\frac{r}{2}}(\bar{x})$  and

$$u^*(t_0) = x_0 + \delta e_n(x_0), \quad \delta = |u^*(t_0) - x_0|.$$

For  $k = \frac{8}{3}\sqrt{2}$  let  $Q_0$  be the set

$$Q_0 = \{x(x_0, (\xi, s)) : |\xi|^2 + (s - \delta)^2 < k^2\delta^2, s > \delta/2\}.$$

Since  $\delta \rightarrow 0$  as  $j_0 \rightarrow +\infty$  we can assume that  $\delta > 0$  is so small ( $\delta < \min\{\frac{1}{2Ck^2}, \frac{r}{1+k}\}$  suffices) that  $\overline{Q_0} \subset \Omega \cap B_r(x_0)$ .

*Claim 1.*  $u^*(t)$  leaves  $\overline{Q_0}$  through the disc  $D_0 = \partial Q_0 \setminus \partial B_{k\delta}(u^*(t_0))$ .

From (2.4) we have  $a_0 \leq \mathcal{A}(v, (T_-, T_+^v))$  for each  $W^{1,2}$  map  $v : (T_-, T_+^v] \rightarrow \mathbb{R}^n$  that coincides with  $u^*$  for  $t \leq t_0$ , and satisfies  $v((t_0, T_+^v)) \subset \Omega$ ,  $v(T_+^v) \in \partial\Omega$  and (1.4). Therefore if we set

$$w(s) = x_0 + s e_n(x_0),$$

$s \in [0, \delta]$ , we have

$$a_0 \leq \mathcal{A}(u^*, (T_-, t_0)) + \mathcal{J}(w, (0, \delta)). \quad (2.18)$$

On the other hand, if  $u^*(t'_0) \in \partial Q_0(x_0) \cap \partial B_{k\delta}(u^*(t_0))$ , where

$$t'_0 = \sup\{t > t_0 : u^*([t_0, t]) \subset \overline{Q_0} \setminus \partial B_{k\delta}(u^*(t_0))\},$$

from (2.7) it follows

$$\mathcal{A}(u^*, (T_-, t_0)) + \mathcal{J}(u^*, (t_0, t'_0)) \leq a_0. \quad (2.19)$$

Using (2.17) we obtain

$$\mathcal{J}(w, (0, \delta)) \leq \frac{4}{3} c^{\frac{1}{2}} \delta^{\frac{3}{2}}, \quad (2.20)$$

and, since

$$c \frac{\delta}{4} \leq U(x(x_0, (\xi, s))), \quad (\xi, s) \in \overline{Q_0}(x_0),$$

we also have, with  $k$  defined above,

$$\frac{8}{3}c^{\frac{1}{2}}\delta^{\frac{3}{2}} = \frac{k}{\sqrt{2}}c^{\frac{1}{2}}\delta^{\frac{3}{2}} \leq \frac{c^{\frac{1}{2}}\delta^{\frac{1}{2}}}{\sqrt{2}} \int_{t_0}^{t'_0} |\dot{u}^*(t)| dt \leq \sqrt{2} \int_{t_0}^{t'_0} \sqrt{U(u^*(t))} |\dot{u}^*(t)| dt. \quad (2.21)$$

From (2.20) and (2.21) it follows

$$\mathcal{J}(w, (0, \delta)) \leq \frac{1}{2} \mathcal{J}(u^*, (t_0, t'_0)),$$

and therefore (2.18) and (2.19) imply the absurd inequality  $a_0 < a_0$ . This contradiction proves the claim.

From Claim 1 it follows that there is  $t_1 \in (t_0, +\infty)$  with the following properties:

$$\begin{aligned} u^*([t_0, t_1]) &\subset Q_0(x_0), \\ u(t_1) &\in D_0. \end{aligned}$$

Set  $x_{0,1} = \pi(u^*(t_1))$  and  $\delta_1 = |u^*(t_1) - x_{0,1}|$ . Since  $h(x_0, 0) = h_\xi(x_0, 0) = 0$  and the radius  $\rho_\delta = (k^2 - \frac{1}{4})^{\frac{1}{2}}\delta$  of  $D_0$  is proportional to  $\delta$ , we can assume that  $\delta$  is so small that the ratio  $\frac{2\delta_1}{\delta}$  and  $\frac{|x_{0,1} - x_0|}{|u^*(t_1) - x(x_0, (0, \frac{\delta}{2}))|}$  are near 1 so that we have

$$\begin{aligned} \delta_1 &\leq \rho\delta, \text{ for some } \rho < 1, \\ |x_{0,1} - x_0| &\leq k\delta. \end{aligned}$$

We also have

$$t_1 - t_0 \leq k'\delta^{\frac{1}{2}}, \quad k' = \frac{8k}{c^{\frac{1}{2}}}.$$

This follows from

$$\begin{aligned} (t_1 - t_0) \frac{c}{4} \delta &\leq \mathcal{A}(u^*, (t_0, t_1)) = \mathcal{J}(u^*, (t_0, t_1)) \\ &= \sqrt{2} \int_{t_0}^{t_1} \sqrt{U(u^*(t))} |\dot{u}^*(t)| dt \leq 2\sqrt{c\delta} |u^*(t_1) - u^*(t_0)| \leq 2c^{\frac{1}{2}}k\delta^{\frac{3}{2}}. \end{aligned}$$

where we used (2.17) to estimate  $\mathcal{J}$  on the segment joining  $u^*(t_0)$  with  $u^*(t_1)$ .

We have  $u^*(t_1) = x_{0,1} + \delta_1 e_n(x_{0,1})$  and we can apply Claim 1 to deduce that there exists  $t_2 > t_1$  such that

$$\begin{aligned} u^*([t_1, t_2]) &\subset Q_1(x_{0,1}), \\ u^*(t_2) &\in D_1, \end{aligned}$$

where  $Q_1$  and  $D_1$  are defined as  $Q_0$  and  $D_0$  with  $\delta_1$  and  $x(x_{0,1}, (\xi, s))$  instead of  $\delta$  and  $x(x_0, (\xi, s))$ . Therefore an induction argument yields sequences  $\{t_j\}$ ,  $\{x_{0,j}\}$ ,  $\{\delta_j\}$  and  $\{Q_j(x_{0,j})\}$  such that

$$\begin{aligned} u^*([t_j, t_{j+1}]) &\subset Q_j(x_{0,j}), \quad x_{0,j} = \pi(u^*(t_j)), \\ \delta_{j+1} &\leq \rho\delta_j \leq \rho^{j+1}\delta, \\ |x_{0,j+1} - x_{0,j}| &\leq k\delta_j \leq k\rho^j\delta, \\ (t_{j+1} - t_j) &\leq k'\delta_j^{1/2} \leq k'\rho^{j/2}\delta^{1/2}, \\ u^*(t_j) &= x_{0,j} + \delta_j e_n(x_{0,j}) \in D_j. \end{aligned} \quad (2.22)$$

We can also assume that  $Q_j(x_{0,j}) \subset \Omega \cap B_r(x_0)$ , for all  $j \in \mathbb{N}$ . This follows from  $|u^*(t_{j+1}) - u^*(t_j)| \leq k\delta_j \leq k\rho^j\delta$ .

From (2.22) we obtain that there exists  $T$  with  $t_0 < T \leq \frac{k'\delta^{\frac{1}{2}}}{1-\rho^{\frac{1}{2}}}$  such that

$$\begin{aligned} u^*(T) &= \lim_{t \rightarrow T} u^*(t) = \lim_{j \rightarrow +\infty} x_{0,j} \in \partial\Omega \setminus P, \\ |u^*(T) - x_0| &\leq \frac{k\delta}{1-\rho}. \end{aligned}$$

This contradicts the existence of the sequence  $\{\tau_j\}$ , with  $\lim_{j \rightarrow \infty} \tau_j = +\infty$ , appearing in (2.16) and establishes (2.15). The proof of the lemma is complete.  $\square$

We continue by showing (2.15) contradicts (2.8).

**Lemma 2.6.** *Assume that  $\Gamma_+$  has positive diameter. Then*

$$T_+ < +\infty.$$

*An analogous statement applies to  $\Gamma_-$  and  $T_-$ .*

*Proof.* From Lemma 2.5, if  $T_+ = +\infty$  there exists  $p \in P$  such that  $\lim_{t \rightarrow +\infty} u^*(t) = p$ . We use a local argument to show that this is impossible if  $\Gamma_+$  has positive diameter. By a suitable change of variable we can assume that  $p = 0$  and that, in a neighborhood of  $0 \in \mathbb{R}^n$ ,  $U$  reads

$$U(u) = V(u) + W(u),$$

where  $V$  is the quadratic part of  $U$ :

$$V(u) = \frac{1}{2} \left( - \sum_{i=1}^m \lambda_i^2 u_i^2 + \sum_{i=m+1}^n \lambda_i^2 u_i^2 \right), \quad \lambda_i > 0 \quad (2.23)$$

and  $W$  satisfies,

$$|W(u)| \leq C|u|^3, \quad |W_x(u)| \leq C|u|^2, \quad |W_{xx}(u)| \leq C|u|. \quad (2.24)$$

Consider the Hamiltonian system with

$$H(p, q) = \frac{1}{2}|p|^2 - U(q), \quad p \in \mathbb{R}^n, \quad q \in \Omega \subset \mathbb{R}^n.$$

For this system the origin of  $\mathbb{R}^{2n}$  is an equilibrium point that corresponds to the critical point  $p = 0$  of  $U$ . Set  $D = \text{diag}(-\lambda_1^2, \dots, -\lambda_m^2, \lambda_{m+1}^2, \dots, \lambda_n^2)$ . The eigenvalues of the symplectic matrix

$$\begin{pmatrix} 0 & D \\ I & 0 \end{pmatrix}$$

are

$$\begin{aligned} &-\lambda_i, \quad i = m+1, \dots, n \\ &\lambda_i, \quad i = m+1, \dots, n \\ &\pm i\lambda_i, \quad i = 1, \dots, m. \end{aligned}$$

Let  $(e_1, 0), \dots, (e_n, 0), (0, e_1), \dots, (0, e_n)$  be the basis of  $\mathbb{R}^{2n}$  defined by  $e_j = (\delta_{j1}, \dots, \delta_{jn})$ , where  $\delta_{ji}$  is Kronecker's delta. The stable  $S^s$ , unstable  $S^u$  and center  $S^c$  subspaces invariant under the flow of the linearized Hamiltonian system at  $0 \in \mathbb{R}^{2n}$  are

$$\begin{aligned} S^s &= \text{span}\{(-\lambda_j e_j, e_j)\}_{j=m+1}^n, \\ S^u &= \text{span}\{(\lambda_j e_j, e_j)\}_{j=m+1}^n, \\ S^c &= \text{span}\{(e_j, 0), (0, e_j)\}_{j=1}^m. \end{aligned}$$

From (2.15) and (1.4) we have

$$\lim_{t \rightarrow +\infty} (\dot{u}^*(t), u^*(t)) = 0 \in \mathbb{R}^{2n}.$$

Let  $W^s$  and  $W^u$  be the local stable and unstable manifold and let  $W^c$  be a local center manifold at  $0 \in \mathbb{R}^{2n}$ . From the center manifold theorem [4], [10], there is a constant  $\lambda_0 > 0$  such that, for each solution  $(p(t), q(t))$  that remains in a neighborhood of  $0 \in \mathbb{R}^{2n}$  for positive time, there is a solution  $(p^c(t), q^c(t)) \in W^c$  that satisfies

$$|(p(t), q(t)) - (p^c(t), q^c(t))| = O(e^{-\lambda_0 t}). \quad (2.25)$$

Since  $W^c$  is tangent to  $S^c$  at  $0 \in \mathbb{R}^{2n}$ , the projection  $W_0^c$  on the configuration space is tangent to  $S_0^c = \text{span}\{e_j\}_{j=1}^m$ , which is the projection of  $S^c$  on the configuration space. Therefore, if  $(p^c, q^c) \neq 0$ , given  $\gamma > 0$ , by (2.25) there is  $t_\gamma$  such that  $d(q(t), S_0^c) \leq \gamma|q(t)|$ , for  $t \geq t_\gamma$ . For  $\gamma$  small, this implies that  $q(t) \notin \Omega$  for  $t \geq t_\gamma$ . It follows that  $(p^c, q^c) \equiv 0$  and from (2.25)  $(p(t), q(t))$  converges to zero exponentially. This is possible only if  $(p(t), q(t)) \in W^s$  and, in turn, only if  $q(t) \in W_0^s$ , the projection of  $W^s$  on the configuration space. This argument leads to the conclusion that the trajectory of  $u^*$  in a neighborhood of 0 is of the form

$$u^*(t(s)) = u^*(s) = s\eta + z(s), \quad (2.26)$$

where

$$\eta = \sum_{i=m+1}^n \eta_i e_i$$

is a unit vector<sup>1</sup>,  $s \in [0, s_0]$  for some  $s_0 > 0$ , and  $z(s)$  satisfies

$$z(s) \cdot \eta = 0, \quad |z(s)| \leq c|s|^2, \quad |z'(s)| \leq c|s| \quad (2.27)$$

for a positive constant  $c$ .

We are now in the position of constructing our local perturbation of  $u$ . We first discuss the case  $U = V$ ,  $z(s) = 0$ . We set

$$\bar{u}(s) = s\eta$$

and, in some interval  $[1, s_1]$ , construct a competing map  $\bar{v} : [1, s_1] \rightarrow \mathbb{R}^n$ ,

$$\bar{v} = \bar{u} + ge_1, \quad g : [1, s_1] \rightarrow \mathbb{R},$$

with the following properties:

$$\begin{aligned} V(\bar{v}(1)) &= 0, \\ \bar{v}(s_1) &= \bar{u}(s_1), \\ \mathcal{J}_V(\bar{v}, [1, s_1]) &< \mathcal{J}_V(\bar{u}, [0, s_1]). \end{aligned} \quad (2.28)$$

The basic observation is that, if we move from  $\bar{u}$  in the direction of one of the eigenvectors  $e_1, \dots, e_m$  corresponding to negative eigenvalues of the Hessian of  $V$ , the potential  $V$  decreases and therefore, for each  $s_0 \in (1, s_1)$  we can define the function  $g$  in the interval  $[1, s_0]$  so that

$$\mathcal{J}_V(\bar{u} + ge_1, (1, s_0)) = \mathcal{J}_V(\bar{u}, (1, s_0)). \quad (2.29)$$

Indeed it suffices to impose that  $g : (1, s_0] \rightarrow \mathbb{R}$  satisfies the condition

$$\sqrt{V(\bar{u}(s))} = \sqrt{1 + g'^2(s)} \sqrt{V(\bar{u}(s) + g(s)e_1)}, \quad s \in (1, s_0].$$

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<sup>1</sup>Actually  $\eta$  coincides with one of the eigenvectors of  $U''(0)$ .

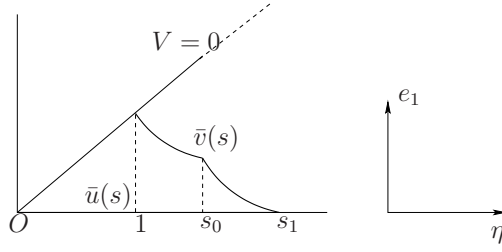


Figure 2: The maps  $\bar{u}(s)$  and  $\bar{v}(s)$ .

According with this condition we take  $g$  as the solution of the problem

$$\begin{cases} g' = -\frac{\lambda_1 g}{\sqrt{s^2 \lambda_\eta^2 - \lambda_1^2 g^2}} = -\frac{\frac{\lambda_1 g}{s \lambda_\eta}}{\sqrt{1 - \frac{\lambda_1^2 g^2}{s^2 \lambda_\eta^2}}}, \\ g(1) = \frac{\lambda_\eta}{\lambda_1} \end{cases}, \quad (2.30)$$

where we have used (2.23) and set

$$\lambda_\eta = \sqrt{\sum_{i=m+1}^n \lambda_i^2 \eta_i^2}.$$

Note that the initial condition in (2.30) implies  $V(\bar{v}(1)) = 0$ . The solution  $g$  of (2.30) is well defined in spite of the fact that the right hand side tends to  $-\infty$  as  $s \rightarrow 1$ . Since  $g$  defined by (2.30) is positive for  $s \in [1, +\infty)$ , to satisfy the condition  $\bar{v}(s_1) = \bar{u}(s_1)$ , we give a suitable definition of  $g$  in the interval  $[s_0, s_1]$  in order that  $g(s_1) = 0$ . Choose a number  $\alpha \in (0, 1)$  and extend  $g$  with continuity to the interval  $[s_0, s_1]$  by imposing that

$$\sqrt{V(\bar{u}(s))} = \alpha \sqrt{1 + g'^2(s)} \sqrt{V(\bar{u}(s) + g(s)e_1)}, \quad s \in (s_0, s_1]. \quad (2.31)$$

Therefore, in the interval  $(s_0, s_1]$ , we define  $g$  by

$$g' = -\frac{1}{\alpha} \sqrt{\frac{1 - \alpha^2 + \alpha^2 \frac{\lambda_1^2 g^2}{s^2 \lambda_\eta^2}}{1 - \frac{\lambda_1^2 g^2}{s^2 \lambda_\eta^2}}} \leq -\frac{\sqrt{1 - \alpha^2}}{\alpha}. \quad (2.32)$$

Since (2.31) implies

$$\mathcal{J}_V(\bar{v}, [s_0, s_1]) = \frac{1}{\alpha} \mathcal{J}_V(\bar{u}, [s_0, s_1]),$$

from (2.29) we see that  $\bar{v}$  satisfies also the requirement (2.28) above if we can choose  $\alpha \in (0, 1)$  and  $1 < s_0 < s_1$  in such a way that

$$\mathcal{J}_V(\bar{u}, (0, 1)) > \frac{1 - \alpha}{\alpha} \mathcal{J}_V(\bar{u}, (s_0, s_1)).$$

Since (2.32) implies  $s_1 < s_0 + \frac{\alpha g(s_0)}{\sqrt{1 - \alpha^2}}$  a sufficient condition for this is

$$\mathcal{J}_V(\bar{u}, (0, 1)) > \frac{1 - \alpha}{\alpha} \mathcal{J}_V\left(\bar{u}, \left(s_0, s_0 + \frac{\alpha g(s_0)}{\sqrt{1 - \alpha^2}}\right)\right),$$

or equivalently

$$1 > \frac{1-\alpha}{\alpha} \left( \left( s_0 + \frac{\alpha g(s_0)}{\sqrt{1-\alpha^2}} \right)^2 - s_0^2 \right) = 2s_0 g(s_0) \sqrt{\frac{1-\alpha}{1+\alpha}} + \frac{\alpha g^2(s_0)}{1+\alpha}. \quad (2.33)$$

By a proper choice of  $s_0$  and  $\alpha$  the right hand side of (2.33) can be made as small as we like. For instance we can fix  $s_0$  so that  $g(s_0) \leq \frac{1}{4}$  and then choose  $\alpha$  in such a way that  $\frac{1}{2}s_0 \sqrt{\frac{1-\alpha}{1+\alpha}} \leq \frac{1}{4}$  and conclude that (2.28) holds.

Next we use the function  $g$  to define a comparison map  $v$  that coincides with  $u^*$  outside an  $\epsilon$ -neighborhood of 0 and show that the assumption that the trajectory of  $u^*$  ends up in some  $p \in P$  must be rejected. For small  $\epsilon > 0$  we define

$$v(\epsilon s) = \epsilon s \eta + z(\epsilon s) + \epsilon g(s - \sigma) e_1, \quad s \in [1 + \sigma, s_1 + \sigma], \quad (2.34)$$

where  $\sigma = \sigma(\epsilon)$  is determined by the condition

$$U(v(\epsilon(1 + \sigma))) = 0,$$

which, using (2.23), (2.24), (2.27) and  $g(1) = \frac{\lambda_\eta}{\lambda_1}$ , after dividing by  $\epsilon^2$ , becomes

$$\frac{1}{2} \lambda_\eta^2 ((1 + \sigma)^2 - 1) = \epsilon f(\sigma, \epsilon), \quad (2.35)$$

where  $f(\sigma, \epsilon)$  is a smooth bounded function defined in a neighborhood of  $(0, 0)$ . For small  $\epsilon > 0$ , there is a unique solution  $\sigma(\epsilon) = O(\epsilon)$  of (2.35). Note also that (2.34) implies that

$$v(\epsilon(s_1 + \sigma)) = u^*(\epsilon(s_1 + \sigma)).$$

We now conclude by showing that, for  $\epsilon > 0$  small, it results

$$\mathcal{J}_U(u^*(\epsilon \cdot), (0, s_1 + \sigma)) > \mathcal{J}_U(v(\epsilon \cdot), (1 + \sigma, s_1 + \sigma)). \quad (2.36)$$

From (2.26) and (2.34) we have

$$\lim_{\epsilon \rightarrow 0^+} \epsilon^{-1} \left| \frac{d}{ds} u^*(\epsilon s) \right| = 1, \quad \lim_{\epsilon \rightarrow 0^+} \epsilon^{-1} \left| \frac{d}{ds} v(\epsilon s) \right| = \sqrt{1 + g'^2(s)}, \quad (2.37)$$

and, using also (2.24) and  $\sigma = O(\epsilon)$ ,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \epsilon^{-2} U(u^*(\epsilon s)) &= V(\bar{u}(s)), \quad s \in (0, s_1), \\ \lim_{\epsilon \rightarrow 0^+} \epsilon^{-2} U(v(\epsilon s)) &= V(\bar{v}(s)), \quad s \in (1, s_1) \end{aligned} \quad (2.38)$$

uniformly in compact intervals.

The limits (2.37) and (2.38) imply

$$\begin{aligned} \lim_{\epsilon \rightarrow 0^+} \epsilon^{-2} \mathcal{J}_U(u^*(\epsilon \cdot), (0, s_1 + \sigma)) &= \lim_{\epsilon \rightarrow 0^+} \sqrt{2} \int_0^{s_1 + \sigma} \sqrt{\epsilon^{-2} U(u^*(\epsilon s))} \epsilon^{-1} \left| \frac{d}{ds} u^*(\epsilon s) \right| ds, \\ &= \sqrt{2} \int_0^{s_1} \sqrt{V(\bar{u}(s))} ds = \mathcal{J}_V(\bar{u}, (0, s_1)) \\ \lim_{\epsilon \rightarrow 0^+} \epsilon^{-2} \mathcal{J}_U(v(\epsilon \cdot), (1 + \sigma, s_1 + \sigma)) &= \lim_{\epsilon \rightarrow 0^+} \sqrt{2} \int_{1 + \sigma}^{s_1 + \sigma} \sqrt{\epsilon^{-2} U(v(\epsilon s))} \epsilon^{-1} \left| \frac{d}{ds} v(\epsilon s) \right| ds, \\ &= \sqrt{2} \int_1^{s_1} \sqrt{V(\bar{v}(s))} \sqrt{1 + g'^2(s)} ds = \mathcal{J}_V(\bar{v}, (1, s_1)). \end{aligned}$$

This and (iii) above imply that, indeed, the inequality (2.36) holds for small  $\epsilon > 0$ . The proof is complete.  $\square$

We can now complete the proof of Theorem 1.1. We show that the map  $u^* : (T_-, T_+) \rightarrow \mathbb{R}^n$  possesses all the required properties. The fact that  $u^*$  satisfies (1.2) and (1.4) follows from Lemma 2.3. Lemma 2.2 implies (1.5) and, if  $T_- > -\infty$ , also (1.6). The fact that  $x_- \in \Gamma_- \setminus P$  is a consequence of Lemma 2.4 and implies that  $\Gamma_-$  has positive diameter. Viceversa, if  $\Gamma_-$  has positive diameter, Lemmas 2.5 and 2.6 imply that  $T_- > -\infty$  and that (1.6) holds for some  $x_- \in \Gamma_- \setminus P$ . The proof of Theorem 1.1 is complete.

*Remark.* From Theorem 1.1 it follows that if  $N$  is even then there are at least  $N/2$  distinct orbits connecting different elements of  $\{\Gamma_1, \dots, \Gamma_N\}$ . If  $N$  is odd there are at least  $(N+1)/2$ . Simple examples show that, given distinct  $\Gamma_i, \Gamma_j \in \{\Gamma_1, \dots, \Gamma_N\}$ , an orbit connecting them does not always exist. Let

$$\mathcal{U}_{ij} = \{u \in W^{1,2}((T_-^u, T_+^u); \mathbb{R}^n) : u((T_-^u, T_+^u)) \subset \Omega, u(T_-^u) \in \Gamma_i, u(T_+^u) \in \Gamma_j\}$$

with  $i \neq j$  and

$$d_{ij} = \inf_{u \in \mathcal{U}_{ij}} \mathcal{A}(u, (T_-^u, T_+^u)).$$

An orbit connecting  $\Gamma_i$  and  $\Gamma_j$  exists if

$$d_{ij} < d_{ik} + d_{kj}, \quad \forall k \neq i, j.$$

The proof of Theorem 1.2 uses, with obvious modifications, the same arguments as in the proof of Theorem 1.1 to characterize  $u^*$  as the limit of a minimizing sequence  $\{u_j\}$  of the action functional

$$\mathcal{A}(u, (0, T^u)) = \int_0^{T^u} \left( \frac{1}{2} |\dot{u}(t)|^2 + U(u(t)) \right) dt. \quad (2.39)$$

in the set

$$\mathcal{U} = \{u \in W^{1,2}((0, T^u); \mathbb{R}^n) : 0 < T_+^u < +\infty, u(0) = 0, u([0, T_+^u)) \subset \Omega, u(T_+^u) \in \partial\Omega\}. \quad (2.40)$$

*Remark.* In the symmetric case of Theorem 1.2 it is easy to construct an example with  $T_+ < T_+^\infty$ . For  $U(x) = 1 - |x|^2$ ,  $x \in \mathbb{R}^2$ , the solution  $u : [0, \pi/2] \rightarrow \mathbb{R}^2$  of (1.2) determined by (1.4) and  $u([0, \pi/2]) = \{(s, 0) : s \in [0, 1]\}$  is a minimizer of  $\mathcal{A}$  in  $\mathcal{U}$ . For  $\epsilon$  small, let  $t_\epsilon = \arcsin(1 - \epsilon)$  and define  $u_\epsilon : [0, T^{u_\epsilon}] \rightarrow \mathbb{R}^2$  as the map determined by (1.4),  $u_\epsilon([0, t_\epsilon]) = \{(s, 0) : s \in [0, 1 - \epsilon]\}$  and  $u_\epsilon((t_\epsilon, T^{u_\epsilon}]) = \{(1 - \epsilon, s) : s \in (0, \sqrt{2\epsilon - \epsilon^2}]\}$ . In this case  $T_+ = \pi/2$  and  $T_+^\infty = 3\pi/4$ .

## 2.1 On the existence of heteroclinic connections

Corollary 1.3 states the existence of heteroclinic connections under the assumptions of Theorem 1.1 and, in particular, that  $U \in C^2$ . Actually, by examining the proof of Theorem 1.1 we can establish an existence result under weaker hypotheses. In the special case  $\partial\Omega = P$ ,  $\#P \geq 2$ , given  $p_- \in P$ , the set  $\mathcal{U}$  defined in (2.1) takes the form

$$\begin{aligned} \mathcal{U} = \{ & u \in W^{1,2}((T_-^u, T_+^u); \mathbb{R}^n) : -\infty < T_-^u < T_+^u < +\infty, \\ & u((T_-^u, T_+^u)) \subset \Omega, U(u(0)) = U_0, u(T_-^u) = p_-, u(T_+^u) \in P \setminus \{p_-\} \}. \end{aligned}$$

In this section we slightly enlarge the set  $\mathcal{U}$  by allowing  $T_\pm^u = \pm\infty$  and consider the admissible set

$$\begin{aligned} \tilde{\mathcal{U}} = \{ & u \in W_{loc}^{1,2}((T_-^u, T_+^u); \mathbb{R}^n) : -\infty \leq T_-^u < T_+^u \leq +\infty, \\ & u((T_-^u, T_+^u)) \subset \Omega, U(u(0)) = U_0, \lim_{t \rightarrow T_-^u} u(t) = p_-, \lim_{t \rightarrow T_+^u} u(t) \in P \setminus \{p_-\} \}. \end{aligned}$$



**Proposition 2.7.** Assume that  $U$  is a non-negative continuous function, which vanishes in a finite set  $P$ ,  $\#P \geq 2$ , and satisfies

$$\sqrt{U(x)} \geq \sigma(|x|), \quad x \in \Omega, \quad |x| \geq r_0$$

for some  $r_0 > 0$  and a non-negative function  $\sigma : [r_0, +\infty) \rightarrow \mathbb{R}$  such that  $\int_{r_0}^{+\infty} \sigma(r) dr = +\infty$ .

Given  $p_- \in P$  there is  $p_+ \in P \setminus \{p_-\}$  and a Lipschitz-continuous map  $u^* : (T_-, T_+) \rightarrow \Omega$  that satisfies (1.4) almost everywhere on  $(T_-, T_+)$ ,

$$\lim_{t \rightarrow T_{\pm}} u^*(t) = p_{\pm},$$

and minimizes the action functional  $\mathcal{A}$  on  $\tilde{\mathcal{U}}$ .

*Proof.* We begin by showing that

$$a_0 = \inf_{u \in \mathcal{U}} \mathcal{A} = \inf_{u \in \tilde{\mathcal{U}}} \mathcal{A} = \tilde{a}_0. \quad (2.41)$$

Since  $\mathcal{U} \subset \tilde{\mathcal{U}}$  we have  $a_0 \geq \tilde{a}_0$ . On the other hand arguing as in the proof of Lemma 2.2, if  $T_+ - T_- = +\infty$ , given a small number  $\epsilon > 0$ , we can construct a map  $u_\epsilon \in \mathcal{U}$  that satisfies

$$a_0 \leq \mathcal{A}(u_\epsilon, (T_-^{u_\epsilon}, T_+^{u_\epsilon})) \leq \mathcal{A}(u, (T_-^u, T_+^u)) + \eta_\epsilon$$

where  $\eta_\epsilon \rightarrow 0$  as  $\epsilon \rightarrow 0$ . This implies  $a_0 \leq \tilde{a}_0$  and establishes (2.41). It follows that we can proceed as in the proof of Theorem 1.1 and define  $u^* \in \tilde{\mathcal{U}}$  as the limit of a minimizing sequence  $\{u_j\} \subset \mathcal{U}$ . The arguments in the proof of Lemma 2.2 show that (2.8) holds. It remain to show that  $u^*$  is Lipschitz-continuous. Looking at the proof of Lemma 2.3 we see that the continuity of  $U$  is sufficient for establishing that (1.4) holds almost everywhere on  $(T_-, T_+)$ , and the Lipschitz character of  $u^*$  follows. The proof is complete.  $\square$

*Remark.* Without further information on the behavior of  $U$  in a neighborhood of  $p_{\pm}$  nothing can be said on  $T_{\pm}$  being finite or infinite and it is easy to construct examples to show that all possible combinations are possible. As shown in Lemma 2.4 a sufficient condition for  $T_{\pm} = \pm\infty$  is that, in a neighborhood of  $p = p_{\pm}$ ,  $U(x)$  is bounded by a function of the form  $c|x - p|^2$ ,  $c > 0$ .  $U$  of class  $C^1$  is a sufficient condition in order that  $u^*$  is of class  $C^2$  and satisfies (1.2).

### 3 Examples

In this section we show a few simple applications of Theorems 1.1 and 1.2.

Our first application describes a class of potentials with the property that, in spite of the existence of possibly infinitely many critical values, (1.2) has a nontrivial periodic orbit on any energy level.

**Proposition 3.1.** Assume that  $U : \mathbb{R}^n \rightarrow \mathbb{R}$  satisfies

$$\begin{aligned} U(-x) &= U(x), \quad x \in \mathbb{R}^n, \\ U(0) &= 0, \quad U(x) < 0 \text{ for } x \neq 0, \\ \lim_{|x| \rightarrow \infty} U(x) &= -\infty \end{aligned}$$

Assume moreover that each non zero critical point of  $U$  is hyperbolic with Morse index  $i_m \geq 1$ . Then there is a nontrivial periodic orbit of (1.2) on the energy level  $\frac{1}{2}|\dot{u}|^2 - U(u) = \alpha$  for each  $\alpha > 0$ .

*Proof.* For each  $\alpha > 0$  we set  $\tilde{U} = U(x) + \alpha$  and let  $\Omega \subset \{\tilde{U} > 0\}$  be the connected component that contains the origin.  $\Omega$  is open, nonempty and bounded and, from the assumptions on the properties of the critical points of  $U$ , it follows that  $\partial\Omega$  is connected and contains at most a finite number of critical points. Therefore we are under the assumptions of Corollary 1.6 for the case  $N = 1$  and the existence of the periodic orbit follows.  $\square$

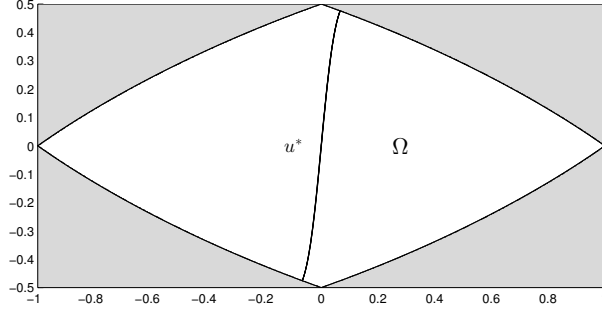


Figure 3: Symmetric periodic orbit for the example with potential (3.1).

An example of potential  $U : \mathbb{R}^2 \rightarrow \mathbb{R}$  that satisfies the assumptions in Proposition 3.1 is, in polar coordinates  $r, \theta$ ,

$$U(r, \theta) = -r^2 + \frac{1}{2} \tanh^4(r) \cos^2(r^{-1}) \cos^{2k}(2\theta),$$

where  $k > 0$  is a sufficiently large number.

Next we give another application of Corollary 1.6. For the potential  $U : \mathbb{R}^2 \rightarrow \mathbb{R}$ , with

$$U(x) = \frac{1}{2}(1 - x_1^2)^2 + \frac{1}{2}(1 - 4x_2^2)^2, \quad (3.1)$$

the energy level  $\alpha = -\frac{1}{2}$  is critical and corresponds to four hyperbolic critical points  $p_1 = (1, 0)$ ,  $-p_1$ ,  $p_2 = (0, \frac{1}{2})$  and  $-p_2$ . The connected component  $\Omega \subset \{\tilde{U} > 0\}$ , ( $\tilde{U} = U(x) - \frac{1}{2}$ ) that contains the origin is bounded by a simple curve  $\Gamma$  that contains  $\pm p_1$  and  $\pm p_2$ . In spite of the presence of these critical points, from Theorem 1.2 it follows that there is a minimizer  $u \in \mathcal{U}$ , with  $\mathcal{U}$  as in (2.40) and  $u(T^u) \in \Gamma \setminus \{\pm p_1, \pm p_2\}$ , and Corollary 1.6 implies the existence of a periodic solution  $v^*$ . Note that there are also two heteroclinic orbits, solutions of (1.2) and (1.4):

$$u_1(t) = (\tanh(t), 0), \quad u_2(t) = (0, \frac{1}{2} \tanh(2t)).$$

These orbits connect  $p_j$  to  $-p_j$ , for  $j = 1, 2$ . By Theorem 1.2 both  $u_1$  and  $u_2$  have action greater than  $v^*|_{(-T_+, T_+)}$ .

Our last example shows that Theorems 1.1 and 1.2 can be used to derive information on the rich dynamics that (1.2) can exhibit when  $U$  undergoes a small perturbation. We consider a family of potentials  $U : \mathbb{R}^2 \times [0, 1] \rightarrow \mathbb{R}$ . We assume that  $U(x, 0) = x_1^6 + x_2^2$  which from various points of view is a structurally unstable potential and, for  $\lambda > 0$  small, we consider the perturbed potential

$$U(x, \lambda) = 2\lambda^4 x_1^2 + x_2^2 - 2\lambda^2 x_1 x_2 - 3\lambda^2 x_1^4 + x_1^6. \quad (3.2)$$

This potential satisfies  $U(-x, \lambda) = U(x, \lambda)$  and, for  $\lambda > 0$ , has the five critical points  $p_0, \pm p_1$  and  $\pm p_2$  defined by

$$\begin{aligned} p_0 &= (0, 0), \\ p_1 &= (\lambda(1 - (\frac{2}{3})^{\frac{1}{2}})^{\frac{1}{2}}, \lambda^3(1 - (\frac{2}{3})^{\frac{1}{2}})^{\frac{1}{2}}), \\ p_2 &= (\lambda(1 + (\frac{2}{3})^{\frac{1}{2}})^{\frac{1}{2}}, \lambda^3(1 + (\frac{2}{3})^{\frac{1}{2}})^{\frac{1}{2}}), \end{aligned}$$

which are all hyperbolic.

We have  $U(p_2, \lambda) < 0 = U(p_0, \lambda) < U(p_1, \lambda)$  and  $p_0$  is a local minimum,  $p_1$  a saddle and  $p_2$  a global minimum. Let  $\alpha$  be the energy level. For  $-\alpha < U(p_2, \lambda)$  or  $-\alpha \geq U(p_1, \lambda)$  no information can be

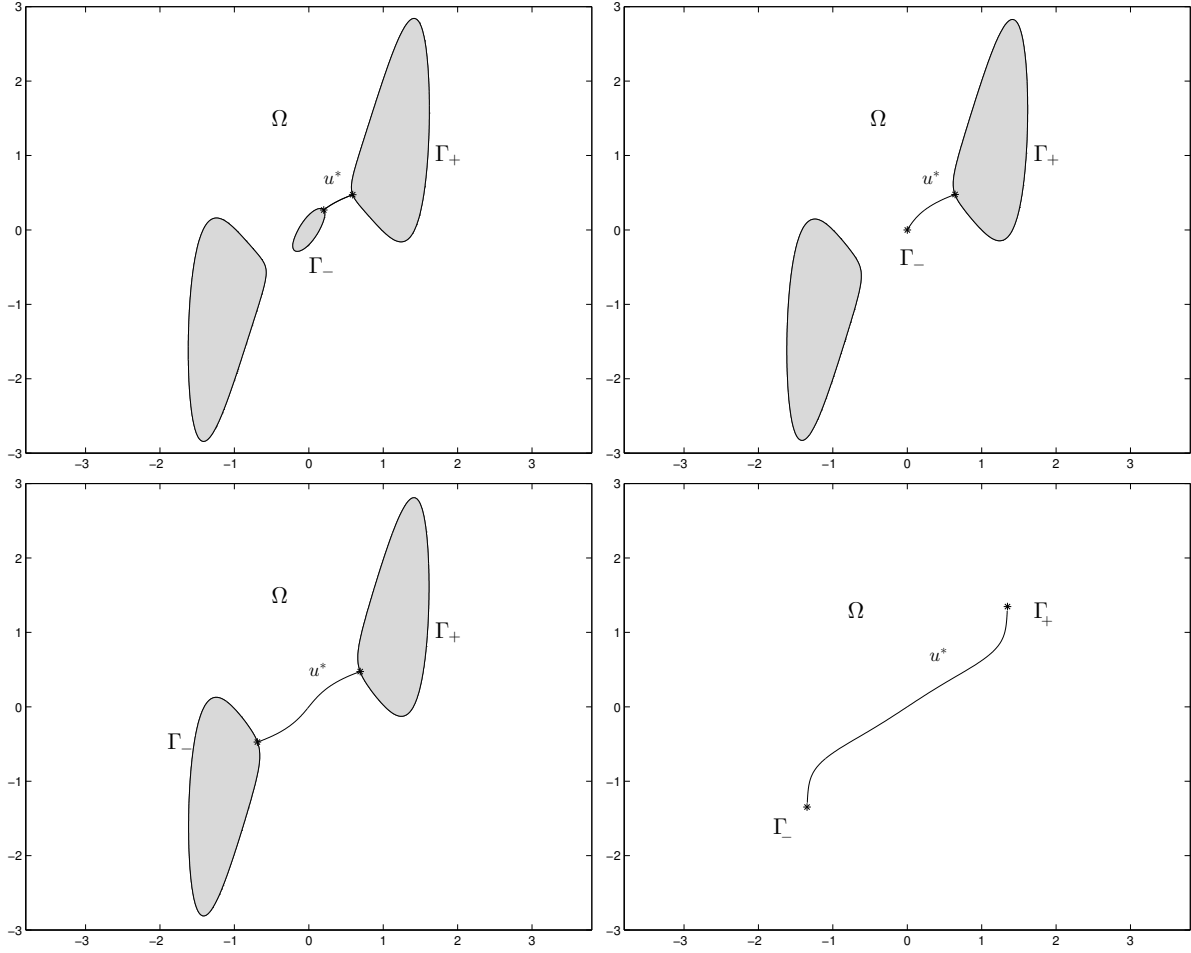


Figure 4: Bifurcations of dynamics of (1.2) with the  $\alpha = 0$ , bottom left:  $\alpha = 0.05$ , bottom right:  $\alpha = -U(p_2, 1)$ . The shaded regions are not accessible.

derived from Theorems 1.1 and 1.2 therefore we assume  $-\alpha \in [U(p_2, \lambda), U(p_1, \lambda))$ . For  $-\alpha = U(p_2, \lambda)$  Corollary 1.3 or Corollary 1.6 yields the existence of a heteroclinic connection  $u_2$  between  $-p_2$  and  $p_2$ . For  $-\alpha \in (U(p_2, \lambda), 0)$  Corollary 1.6 implies the existence of a periodic orbit  $u_\alpha$ . This periodic orbit converges uniformly in compact intervals to  $u_2$  and the period  $T_\alpha \rightarrow +\infty$  as  $-\alpha \rightarrow U(p_2, \lambda)^+$ . For  $\alpha = 0$  Corollary 1.4 implies the existence of two orbits  $u_0$  and  $-u_0$  homoclinic to  $p_0 = 0$ . We can assume that  $u_0$  satisfies the condition  $u_0(-t) = u_0(t)$  and that  $u_\alpha(0) = 0$ . Then we have that  $u_\alpha(\cdot \pm \frac{T_\alpha}{4})$  converges uniformly in compact intervals to  $\mp u_0$  and  $T_\alpha \rightarrow +\infty$  as  $-\alpha \rightarrow 0^-$ . For  $-\alpha \in (0, U(p_1, \lambda))$ ,  $\partial\Omega$  is the union of three simple curves all of positive diameter:  $\Gamma_0$  that includes the origin and  $\pm\Gamma_2$  which includes  $\pm p_2$  and Corollary 1.5 together with the fact that  $U(\cdot, \lambda)$  is symmetric imply the existence of two periodic solutions  $\tilde{u}_\alpha$  and  $-\tilde{u}_\alpha$  with  $\tilde{u}_\alpha$  that oscillates between  $\Gamma_0$  and  $\Gamma_2$  in each time interval equal to  $\frac{T_\alpha}{2}$ . Assuming that  $\tilde{u}_\alpha(0) \in \Gamma_2$  we have that, as  $-\alpha \rightarrow 0^+$ ,  $\tilde{u}_\alpha \rightarrow u_0$  uniformly in compacts and  $T_\alpha \rightarrow +\infty$ . Finally we observe that, in the limit  $-\alpha \rightarrow U(p_1, \lambda)^-$ ,  $\tilde{u}_\alpha$  converges uniformly in  $\mathbb{R}$  to the constant solution  $u \equiv p_1$ .

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